JOURNAL OF APPROXIMATION THEORY **41**, 39–50 (1984)

Some Weighted Polynomial Inequalities*

R. A. ZALIK

Department of Mathematics, Auburn University, Alabama 36849, USA

Communicated by Paul G. Nevai

Received February 22, 1983; revised July 5, 1983

We shall use the following notation: x will always denote a variable defined on $(-\infty, \infty)$, y will always denote a variable defined on $(0, \infty)$, $\|f(x)\|_p$ will stand for the norm of f in $L_p(-\infty, \infty)$ and $\|f(y)\|_p$ for the norm of f in $L_p(0, \infty)$; for $\beta \geqslant 0$, $W_\beta(x) = (1 + x^2)^{\beta/2} \exp(-x^2/2)$ and $V_\beta(y) = (1 + y)^{\beta/2} \exp(-y/2)$; n will denote a strictly positive integer, and q_n an arbitrary polynomial of degree at most n; by c we shall denote positive numbers depending at most on β, and by $c(\cdot)$ positive numbers depending at most on β and on the variables enclosed by the parentheses, but not necessarily the same positive number if they appear more than once in the same formula.

This paper is a sequel to [1], and like it has been deeply influenced by the ideas of G. Freud. The first five theorems below present polynomial inequalities on $(-\infty, \infty)$ involving the weight $W_\beta(x)$; the case $\beta = 0$ of these results was proved in [1]. The remaining five theorems present polynomial inequalities on $[0, \infty)$ involving the weight $V_\beta(y)$. The functions $W_\beta(x)$ were introduced by Freud in [2]. Note that if $Q_\beta(x) = -\ln[W_\beta(x)]$ and $\beta > 16[\exp(1/16) - 1] > 1.04$, then $Q_\beta[(\beta/16)^{1/2}] < 0$, and therefore $W_\beta(x)$ does not satisfy one of the hypotheses of [3]. Moreover $Q_\beta''(0) = 1 - \beta$; thus if $\beta > 1$, $W_\beta(x)$ is neither very strongly regular nor superregular in the sense of Mhaskar [4. 5]. Hence the theorems in this paper are not contained in, nor can be trivially inferred from, the results of these authors.

We start with:

THEOREM 1. *Let $0 \leqslant r < \infty$ and $1 \leqslant p \leqslant \infty$. Then*

$$\| |x|^r W_\beta(x)\, q_n(x)\|_p \leqslant c(m)\, \| |x|^r W_\beta(x)\, q_n(x)\|_{L_p(-4\sqrt{n}, 4\sqrt{n})},$$

where $m - 1$ is the integral part of $r + \beta$.

* Dedicated to Dr. Emilie V. Haynsworth on the occasion of her retirement.

0021-9045/84 $3.00

Proof. Since

$$|x|^{\beta} W_0(x) \leqslant W_\beta(x) \leqslant 2^{\beta/2}(1 + |x|^{\beta})\, W_0(x), \tag{1}$$

the case $p = \infty$ follows by combining the cases $\rho = r$ and $\rho = r + \beta$ of [6, Lemma 3]. Assume therefore that $p < \infty$, and let $I_n = \{x/|x| \geqslant 4\sqrt{n}\}$, $J_n = \{x/4\sqrt{n} \leqslant |x| < 4\sqrt{n+m}\}$, and $V_n = \{x/|x| \geqslant 4\sqrt{n+m}\}$. If $A_n = [\int_{I_n} ||x|^r W_\beta(x)\, q_n(x)|^p\, dx]^{1/p}$, applying [1, (12)] and the remarks that follow it, and bearing in mind that if $|x| \geqslant 1$, then $W_\beta(x) \leqslant 2^{\beta/2}\, |x|^{\beta} W_0(x)$, we have

$$\begin{aligned}
(A_n)^p &= \int_{J_n} ||x|^r W_\beta(x)\, q_n(x)|^p\, dx + \int_{V_n} ||x|^r W_\beta(x)\, q_n(x)|^p\, dx \\
&\leqslant 2^{p\beta/2}[16(n+m)]^{p(\beta+r)/2} \int_{J_n} |W_0(x)\, q_n(x)|^p\, dx \\
&\quad + 2^{p\beta/2} \int_{V_n} ||x|^{r+\beta} W_0(x)\, q_n(x)|^p\, dx \\
&\leqslant [c(m)]^p \left[n^{pm/2} \int_{I_n} |W_0(x)\, q_n(x)|^p\, dx + \int_{V_n} |x^m W_0(x)|^p\, dx \right] \\
&\leqslant [nc(m)]^p [n^{pm/2} \exp(-cpn)(\| W_0(x)\, q_n(x)\|_p)^p \\
&\quad + \exp(-cp[n+m])(\| x^m W_0(x)\, q_n(x)\|_p)^p] \\
&\leqslant [nc(m)]^p [n^{pm/2} \exp(-cpn)(\| W_\beta(x)\, q_n(x)\|_p)^p \\
&\quad + \exp(-cp[n+m])(\| x^m W_0(x)\, q_n(x)\|_p)^p].
\end{aligned}$$

An inspection of the proof of [1, Theorem 1] shows that

$$\|x^m W_0(x)\, q_n(x)\|_p \leqslant c(m)\, n^{m/2}\, \| W_0(x)\, q_n(x)\|_p \leqslant c(m)\, n^{m/2}\, \| W_\beta(x)\, q_n(x)\|_p;$$

thus

$$A_n \leqslant c(m)\, n^{(m+1)/2} \exp(-cn)\, \| W_\beta(x)\, q_n(x)\|_p.$$

Since

$$\| |x|^r\, W_\beta(x)\, q_n(x)\|_p \leqslant \| |x|^r\, W_\beta(x)\, q_n(x)\|_{L_p(-4\sqrt{n}, 4\sqrt{n})} + A_n,$$

the conclusion follows. Q.E.D.

For $0 < \rho < \infty$, a result similar to Theorem 1 can be derived from Bonan [6, (3.2.3)]. However, the integral on the right-hand side of the inequality would be defined over an interval with endpoints at $\pm[(1+\lambda)(2n + \beta/2 + 1/2)]^{1/2}$, where λ is any positive real number. Thus the inequality that can be inferred from Bonan's result is superior for small values of β, whereas for large values Theorem 1 is better.

THEOREM 2. (a) *Let* $1 \leqslant p \leqslant \infty$ *and* $0 \leqslant r, \alpha < \infty$. *Then*

$$\| |x|^{r+\alpha} W_\beta(x)\, q_n(x)\|_p \leqslant c(r, \alpha)\, n^{\alpha/2} \, \| |x|^r W_\beta(x)\, q_n(x)\|_p$$

(b) *The above inequality is optimal in the sense that for any choice of* r, α *and* p, $(r, \alpha \geqslant 0;\ 1 \leqslant p \leqslant \infty)$, $c(r, \alpha)$ *cannot be replaced by a sequence* $\{c_n\}$ *that converges to zero as n tends to infinity.*

Proof. (a) Applying Theorem 1 we have

$$\begin{aligned} \| |x|^{r+\alpha} W_\beta(x)\, q_n(x)\|_p &\leqslant c(r+\alpha)\, \| |x|^{r+\alpha} W_\beta(x)\, q_n(x)\|_{L_p(-4\sqrt{n}, 4\sqrt{n})} \\ &\leqslant c(r, \alpha)\, n^{\alpha/2} \, \| |x|^r W_\beta(x)\, q_n(x)\|_{L_p(-4\sqrt{n}, 4\sqrt{n})} \\ &\leqslant c(r, \alpha)\, n^{\alpha/2} \, \| |x|^r W_\beta(x)\, q_n(x)\|_p, \end{aligned}$$

and the conclusion follows.

(b) Proceeding as in the proof of [1, Theorem 1(b)] it is readily seen that for any $\delta \geqslant 0$ and $1 \leqslant p < \infty$,

$$(\| |x|^\delta W_0(x)\|_p)^p = (2/p)^{(1/2)(\delta p + 1)} \Gamma[(1/2)(\delta p + 1)].$$

Let $q_n(x) = x^n$; thus $| |x|^{r+\alpha} W_\beta(x)\, q_n(x)| = |x|^{r+\alpha+n} W_\beta(x)$, and from (1) we infer that if $1 \leqslant p < \infty$,

$$\begin{aligned} \| |x|^{r+\alpha} W_\beta(x)\, q_n(x)\|_p &\geqslant \| |x|^{r+\alpha+\beta+n} W_0(x)\|_p \\ &= (2/p)^{(1/2)[r+\alpha+\beta+n+1/p]} (\Gamma[(r+\alpha+\beta+n)\, p/2 + 1/2]^{1/p}) \end{aligned}$$

and

$$\begin{aligned} \| |x|^r W_\beta(x)\, q_n(x)\|_p &\leqslant 2^{\beta/2} \, \|(1 + |x|^\beta)\, |x|^{r+n} W_0(x)\|_p \\ &\leqslant 2^{\beta/2} (\| |x|^{r+n} W_0(x)\|_p + \| |x|^{r+\beta+n} W_0(x)\|_p) \\ &= c(2/p)^{(1/2)[r+n+1/p]} (\Gamma[(r+n)\, p/2 + 1/2])^{1/p} \\ &\quad + (2/p)^{(1/2)[r+\beta+n+1/p]} (\Gamma[(r+\beta+n)\, p/2 + 1/2])^{1/p}]. \end{aligned}$$

Applying Stirling's formula we therefore see that

$$\| |x|^{r+\alpha} W_\beta(x)\, q_n(x)\|_p \geqslant c(r, \alpha)\, n^{\alpha/2} \, \| |x|^r W_\beta(x)\, q_n(x)\|_p,$$

and the conclusion follows.

We now prove the assertion for $p=\infty$. Using elementary calculus it is easy to see that for any $\delta \geqslant 0$, $\| |x|^{\delta} W_0(x)\|_\infty = \delta^{\delta/2} \exp(-\delta/2)$. Applying (1) we thus have

$$\| |x|^{r-\alpha} W_\beta(x)\, q_n(x)\|_\infty \geqslant (r+\alpha+\beta+n)^{(1/2)(r+\alpha+\beta+n)} \exp[-(1/2)(r+\alpha+\beta+n)]$$

and

$$\begin{aligned}\| |x|^{r} W_\beta(x)\, q_n(x)\|_\infty &\leqslant 2^{\beta/2}(\| |x|^{r+n} W_0(x)\|_\infty + \| |x|^{r+\beta+n} W_0(x)\|_\infty) \\ &= 2^{\beta/2}[(r+n)^{(1/2)(r+n)} \exp[-(1/2)(r+n)] \\ &\quad + (r+\beta+n)^{(1/2)(r+\beta+n)} \exp[-(1/2)(r+\beta+n)]]\end{aligned}$$

whence the conclusion readily follows. Q.E.D.

Part (a) of the following theorem was proved by G. Freud [2, p. 129, Theorem 2]. A particular case appears in [1].

THEOREM 3. (a) *Let $1 \leqslant p \leqslant \infty$; then for any natural number s,*

$$\| [W_\beta(x)\, q_n(x)]^{(s)} \|_p \leqslant c(s)\, n^{s/2}\, \| W_\beta(x)\, q_n(x)\|_p.$$

(b) *The above inequality is optimal (in the sense of Theorem 2).*

Proof of (b). For the purposes of this proof we shall say that $a_n \approx b_n$ if there are two constants $K_1(k)$ and $K_2(k)$ such that $K_1(k)\,|b_n| \leqslant |a_n| \leqslant K_2(k)\,|b_n|$. Let $H_n(x)$ denote the nth Hermite polynomial; from [7, p. 838, 7.375-1] and Stirling's formula

$$\int_R \exp(-2x^2)\, H_n^2(x)\, H_{2k}(x)\, dx = \pi^{-1} 2^{(n+k-1/2)} [\Gamma(k+1/2)]^2 \Gamma(n-k+1/2) \approx n^{-k} 2^n \Gamma(n+1/2).$$

Since $(1+2x^2)^k = \sum_{r=0}^{k} a_r H_{2r}(x)$ we thus infer that

$$\int_R \exp(-2x^2)\, H_n^2(x)(1+2x^2)^k\, dx \approx \sum_{r=0}^{k} a_r n^{-r} 2^n \Gamma(r+1/2) \approx 2^n \Gamma(n+1/2).$$

From the inequality $(1+2x^2)^k \leqslant (1+2x^2)^\delta \leqslant (1+2x^2)^{k+1}$, where k is the integral part of δ, it is clear that the preceding asymptotic formula is also satisfied if k is replaced by any non-negative real number. Setting $\delta=\beta/2$, $He_n(x) = H_n(x\sqrt{2})$ and making the change of variable $\sqrt{2}\,x \to x$ we thus see that for any $\beta \geqslant 0$,

$$\| W_\beta(x)\, He_n(x)\|_2 \approx [2^n \Gamma(n+1/2)]^{1/2}. \tag{2}$$

Since $[W_0(x)\, He_n(x)]^{(r)} = (-1)^r 2^{-r/2} W_0(x)\, He_{n+r}(x)$, applying the Leibnitz rule we have

$$\begin{aligned}[W_\beta(x)\, He_n(x)]^{(s)} &= [(1+x^2)^{\beta/2} W_0(x)\, He_n(x)]^{(s)} \\ &= \sum_{r=0}^{s-1} \{c(r,s)[(1+x^2)^{\beta/2}]^{(s-r)} W_0(x)\, He_{n+r}(x)\} \\ &\quad + (-1)^s 2^{-s} W_\beta(x)\, He_{n+s}(x).\end{aligned}$$

Thus, since $|[(1+x^2)^{\beta/2}]^{(s-r)}| \leqslant c(r)(1+x^2)^{\beta/2}$,

$$\begin{aligned}\| W_\beta(x)\, He_{n+s}(x)\|_2 \leqslant c(s)\, \| [W_\beta(x)\, He_n(x)]^{(s)} \|_2 \\ + \sum_{r=0}^{s-1} c(r,s)\, \| W_\beta(x)\, He_{n+r}(x)\|_2.\end{aligned}$$

Since $\Gamma(x+1) = x\Gamma(x)$, we infer from (2) that if $r < s$, $\| W_\beta(x)\, He_{n+r}(x)\|_2 = a_n(r)\, \| W_\beta(x)\, He_{n+s}(x)\|$, where $\lim_{n\to\infty} a_n(r) = 0$; hence the preceding inequality implies that $\| W_\beta(x)\, He_{n+s}(x)\|_2 \leqslant c(s)\, \| [W_\beta(x)\, He_n(x)]^{(s)} \|_2$. Since (2) also implies that $\| W_\beta(x)\, He_n(x)\|_2 \leqslant c(s)\, n^{-s/2}\, \| W_\beta(x)\, He_{n+s}(x)\|_2$, we conclude that $\| W_\beta(x)\, He_n(x)\|_2 \leqslant c(s)\, n^{-s/2}\, \| [W_\beta(x)\, He_n(x)]^{(s)} \|_2$, which proves the assertion for $p = 2$.

We shall now prove the statement for every $p \geqslant 1$. Let $f(x) \sim \sum a_r(f)\, p_r(W_\beta^2; x)$ be the expansion of $f(x)$ in the polynomials $p_r(W_\beta^2; x)$ orthogonal with respect to the weight $W_\beta^2(x)$ on $(-\infty, \infty)$, let $s_m(W_\beta^2; f; x)$ be the sum of the terms $r \leqslant m$ of this expansion, and let $V_{2n}(W_\beta^2; f; x) = n^{-1} \sum_{m=n+1}^{2n} s_m(W_\beta^2; f; x)$. From [2, (39), (40)] or [8, Lemma 2.6], and the Riesz–Thorin theorem [9, Vol. 2, p. 95], we readily infer that for all p such that $1 \leqslant p \leqslant \infty$ and every measurable function $f(x)$ such that $\| W_\beta(x)\, f(x)\|_p < \infty$,

$$\| W_\beta(x)\, V_{2n}(W_\beta^2; f; x)\|_p \leqslant c(p)\, \| W_\beta(x)\, f(x)\|_p. \tag{3}$$

Assume now that for some s and p and for every polynomial sequence $\{q_n(x)\}$ there is a sequence $\{c_n\}$, converging to zero, such that

$$\| [W_\beta(x)\, q_n(x)]^{(s)} \|_p \leqslant c_n n^{s/2}\, \| W_\beta(x)\, q_n(x)\|_p. \tag{4}$$

Define the linear operator $T_{n,s}(f)$ by $[T_{n,s}(f)](x) = [W_\beta(x)\, V_{2m}(f; x)]^{(s)}$. Applying (4) and then (3) we readily infer that $\| [T_{n,s}(f)](x)\|_p \leqslant c_n c(p)(2n)^{s/2}\, \| W_\beta(x)\, f(x)\|_p$. The argument is completed exactly as in the proof of [1, Theorem 2(b)]. Q.E.D.

THEOREM 4. *Let $r \geqslant 0$ and $1 \leqslant p, p_1 \leqslant \infty$. Then*

$$\| |x|^r\, W_\beta(x)\, q_n(x)\|_p \leqslant c(r)\, n^{|1/(2p) - 1/(2p_1)|}\, \| |x|^r\, W_\beta(x)\, q_n(x)\|_{p_1}.$$

Proof. For $p < p_1 \leqslant \infty$ the proof is identical to that of the corresponding case of [1, Theorem 3], using Theorem 1 instead of [1, (13)].

Let x_0 be such that $\| |x|^r W_\beta(x) q_n(x)\|_\infty = |x_0|^r |W_\beta(x_0) q_n(x_0)|$. We now show that for every real t,

$$|1 - c(r) n^{1/2} |t - x_0|| \, \| |x|^r W_\beta(x) q_n(x)\|_\infty \leqslant | |t|^r W_\beta(t) q_n(t)|, \qquad (5)$$

where $c(r) > 0$. If r is an integer, the proof is identical to that of [1, (23)], using Theorem 3 instead of [1, Theorem 2]. To prove it for other values of r, let k be the integral part of r, and $c(r) = \max\{c(k), c(k+1)\}$. Let t be arbitrary but fixed.

If $1 - c(r) n^{1/2}|t - x_0| \leqslant 0$, (which happens in particular if $t = 0$), (5) is trivial. Assume therefore that $1 - c(r) n^{1/2} |t - x_0| > 0$, and let $h(r) = \| |x/t|^r W_\beta(x) q_n(x)\|_\infty$. Then (5) is equivalent to

$$h(r) \leqslant |W_\beta(t) q_n(t)| \, |1 - c(r) n^{1/2} |t - x_0||^{-1}.$$

Since $1 - c(k) n^{1/2} |t - x_0| > 0$ and $1 - c(k+1) n^{1/2} |t - x_0| > 0$, the preceding inequality is satisfied for $r = k$ and $r = k + 1$, and the conclusion readily follows by noting that since $h(r)$ is convex, $h(r) \leqslant \max\{h(k), h(k+1)\} \leqslant |W_\beta(t) q_n(t)| \, |1 - c(r) n^{1/2} |t - x_0||^{-1}$.

The remainder of the proof is carried out exactly as in the proof of [1, Theorem 3], using (5) instead of [1, (23)]. Q.E.D.

Theorem 4 should also follow from [5, Theorem 2].

A result similar to Theorem 4 can be inferred from Mhaskar and Saff [10, Theorems 6.1 and 6.4]. However, the constant that would appear in the inequality would depend also on p and p_1.

A converse of Theorem 2 is

THEOREM 5. *Let $\alpha, \delta \geqslant 0$ and $1 \leqslant p \leqslant \infty$. Then*

$$\| |x|^\delta W_\beta(x) q_n(x)\|_p \leqslant c(\alpha, \delta) n^{\alpha/2} \| |x|^{\alpha + \delta} W_\beta(x) q_n(x)\|_p.$$

The proof of this assertion is based on the following:

LEMMA. *Let $\delta \geqslant 0$ and $1 \leqslant p \leqslant \infty$. Then*

$$\| |x|^\delta W_\beta(x) q_n(x)\|_p \leqslant c(\delta) n^{1/2} \| |x|^{\delta + 1} W_\beta(x) q_n(x)\|_p.$$

Proof. We first prove the statement for $p = \infty$ and $\beta = 0$. Let $\gamma \geqslant 0$ and $\| |x|^\gamma W_0(x) q_n(x)\|_\infty = |x_0|^\gamma W_0(x_0) |q_n(x_0)|$. Making if necessary a change of variable of the form $x \to -x$, we can assume without essential loss of generality that $x_0 \geqslant 0$.

Setting $f(x)=x^{\gamma+1}W_0(x)\,q_n(x)$ $(x\geqslant 0)$ and applying the mean value theorem we see that if $x_0>0$, $f(x_0)=x_0 f'(\xi)$, where $0<\xi<x_0$. Dividing by x_0 we have $x_0^\gamma W_0(x_0)\,q_n(x_0)=f'(\xi)$, i.e.,

$$\|\,|x|^\gamma W_0(x)\,q_n(x)\|_\infty=|f'(\xi)|. \tag{6}$$

An application of the mean value theorem and a limiting process also show that if $x_0=0$, (6) is satisfied for $\xi=0$. (This case is of significance only if $\gamma=0$.) Let m denote the integral part of γ; then $\gamma=m+\alpha$, where $0\leqslant\alpha<1$. Since $f(x)=x^\alpha W_0(x)[x^{m+1}q_n(x)]$, it is clear that $f'(x)=\alpha x^\gamma W_0(x)\,q_n(x)-x^{\alpha+1}W_0(x)[x^{m+1}q_n(x)]+x^\alpha W_0(x)[x^{m+1}q_n(x)]'$. Thus from (6),

$$\|\,|x|^\gamma W_0(x)\,q_n(x)\|_\infty\leqslant\alpha\,\|\,|x|^\gamma W_0(x)\,q_n(x)\|_\infty+\|\,|x|^{\gamma+2}W_0(x)\,q_n(x)\|_\infty + \|\,|x|^\alpha W_0(x)[x^{m+1}q_n(x)]'\|_\infty,$$

i.e.,

$$(1-\alpha)\,\|\,|x|^\gamma W_0(x)\,q_n(x)\|_\infty \leqslant\|\,|x|^{\gamma+2}W_0(x)\,q_n(x)\|_\infty+\|\,|x|^\alpha W_0(x)[x^{m+1}q_n(x)]'\|_\infty.$$

Since $1-\alpha>0$, applying Theorem 2 to the first term in the right-hand member of the preceding inequality, and [2, (29)] or [11, Theorem 8] to the second term, we see that

$$\|\,|x|^\gamma W_0(x)\,q_n(x)\|_\infty\leqslant c(\gamma)\,n^{1/2}\,\|\,|\gamma|^{\gamma+1}W_0(x)\,q_n(x)\|_\infty.$$

Combining the cases $\gamma=\beta$ and $\gamma=\beta+\delta$ of the preceding inequality with (1) and the inequality $0<W_0(x)\leqslant W_\beta(x)$, we readily conclude that

$$\|\,|x|^\delta W_\beta(x)\,q_n(x)\|_\infty\leqslant c(\delta)\,n^{1/2}\,\|\,|x|^{\delta+1}W_\beta(x)\,q_n(x)\|_\infty, \tag{7}$$

which is the result for $p=\infty$ and $\beta\geqslant 0$.

To prove the assertion for $1\leqslant p<\infty$, let $I_n=[-n^{-1/2},n^{-1/2}]$, and let J_n be the complementary set of I_n in $(-\infty,\infty)$. If x is in J_n, $1\leqslant|x|\,n^{1/2}$; thus

$$\begin{aligned}\int_{J_n}|\,|x|^\delta W_\beta(x)\,q_n(x)|^p\,dx&\leqslant n^{p/2}\int_{J_n}|\,|x|^{\delta+1}W_\beta(x)\,q_n(x)|^p\,dx\\ &\leqslant[n^{1/2}\,\|\,|x|^{\delta+1}W_\beta(x)\,q_n(x)\|_p]^p.\end{aligned} \tag{8}$$

On the other hand, applying the mean value theorem of the integral calculus, (7), and Theorem 4, we have

$$\int_{I_n} ||x| W_\beta(x) q_n(x)|^p \, dx = 2n^{-1/2} ||t|^\delta W_\beta(t) q_n(t)|^p$$
$$\leqslant 2[n^{-1/(2p)} \||x|^\delta W_\beta(x) q_n(x)\|_\infty]^p$$
$$\leqslant [c(\delta) n^{-1/(2p)+1/2} \| |x|^{\delta+1} W_\beta(x) q_n(x)\|_\infty]^p$$
$$\leqslant [c(\delta) n^{1/2} c(\delta) \| |x|^{\delta+1} W_\beta(x) q_n(x)\|_p]^p.$$

Combining the preceding inequality with (8), the conclusion follows. Q.E.D.

Proof of Theorem 5. It suffices to assume that $q_n(x)$ is not identically zero. The assertion is trivial for $\alpha = 0$ and follows by repeated application of the Lemma if α is a natural number. Since the inequality is equivalent to

$$(\| |n^{1/2}x|^\alpha |x|^\delta W_\beta(x) q_n(x)\|_p)^{-1} \| |x|^\delta W_\beta(x) q_n(x)\|_p \leqslant c(\alpha, \delta),$$

and for fixed δ the left-hand member of the preceding expression is a convex function h of α, the conclusion now readily follows by noting that if $m-1$ is the integral part of $2r + \beta + 1/p$, then $h(\alpha) \leqslant \max\{h(0), h(m)\}$. Q.E.D.

A result similar to Theorem 5 can be inferred from [6, p. 26, (3.2.27)] and Mhaskar and Saff [12]. However, the constant c that appears in the inequality would depend also on p.

The remaining results concern the interval $(0, \infty)$:

THEOREM 6. *Let* $r \geqslant 0$ *and* $1 \leqslant p \leqslant \infty$. *Then*

$$\| y^r V_\beta(y) q_n(y)\|_p \leqslant c(r) \| y^r V_\beta(y) q_n(y)\|_{L_p(0, 32n)}.$$

Proof. Assume first that $1 \leqslant p < \infty$. Since $q_n(x^2)$ is an even function, making the change of variable $y = x^2$ and applying Theorem 1 we see that if $m-1$ is the integral part of $2r + \beta + 1/p$,

$$\| y^r V_\beta(y) q_n(y)\|_p = \left[2 \int_0^\infty ||x|^{2r} W_\beta(x) q_n(x^2)|^p x \, dx \right]^{1/p}$$
$$= \| |x|^{2r+1/p} W_\beta(x) q_n(x^2)\|_p$$
$$\leqslant c(m) \| |x|^{2r+1/p} W_\beta(x) q_n(x^2)\|_{L_p(-4\sqrt{2n}, 4\sqrt{2n})}$$
$$= 2^{1/p} c(m) \| |x|^{2r+1/p} W_\beta(x) q_n(x^2)\|_{L_p(0, 4\sqrt{2n})}$$
$$= 2^{1/p} c(m) \left[\int_0^{4\sqrt{2n}} |x^{2r} W_\beta(x) q_n(x^2)|^p x \, dx \right]^{1/p}$$
$$\leqslant c(m) \| y^r V_\beta(y) q_n(y)\|_{L_p(0, 32n)}.$$

The conclusion now follows by noting that if $m_1 - 1$ is the integral part of $2r+\beta$, then $m = m_1$ or $m = m_1 + 1$, whence $c(m) \leqslant \max\{c(m_1), c(m_1+1)\}$ (i.e. $c(m) \leqslant c(r)$). The proof for $p = \infty$ is similar and will be omitted.

Q.E.D.

THEOREM 7. (a) *Let* $1 \leqslant p \leqslant \infty$ *and* $0 \leqslant r, \alpha < \infty$. *Then*

$$\| y^{r+\alpha} V_\beta(y)\, q_n(y)\|_p \leqslant c(r, \alpha)\, n^\alpha \, \| y^r V_\beta(y)\, q_n(y)\|_p.$$

(b) *The preceding inequality is optimal* (*in the sense of Theorem* 2).

Proof. Part (a) is a trivial consequence of Theorem 6. To prove (b), set $q_n(y) = y^n$, and proceed as in the proof of Theorem 2(b), using the fact that for $y \geqslant 0$,

$$y^{\beta/2} V_0(y) \leqslant V_\beta(y) = (1+y)^{\beta/2} V_0(y). \qquad \text{Q.E.D.}$$

THEOREM 8. *Let* $r \geqslant 0$ *and* $1 \leqslant p, p_1 \leqslant \infty$. *Then*

$$\| y^r V_\beta(y)\, q_n(y)\|_p \leqslant c(r)\, n^{|1/p - 1/p_1|} \, \| y^r V_\beta(y)\, q_n(y)\|_{p_1}.$$

Proof. Let $y = x^2$. Since

$$\| y^r V_\beta(y)\, q_n(y)\|_p = \| \, |x|^{2r+1/p} W_\beta(x)\, q_n(x^2)\|_p,$$

the conclusion readily follows from Theorem 4 if we notice that the function $c(r)$ that appears in the statement of Theorem 4 can be taken to be constant between consecutive integers and use an argument similar to the one employed at the end of the proof of Theorem 6 to prove independence from p. Q.E.D.

The following is a trivial consequence of Theorem 5, obtained by the change of variable $y = x^2$.

THEOREM 9. *Let* $1 \leqslant p \leqslant \infty$, $\alpha \geqslant 0$, *and* $\delta \geqslant -1/(2p)$. *Then*

$$\| y^\delta V_\beta(y)\, q_n(y)\|_p \leqslant c(\alpha, \delta)\, n^\alpha \, \| y^{\alpha+\delta} V_\beta(y)\, q_n(y)\|_p.$$

Finally, we have

THEOREM 10. *Let* $r \geqslant 0$ *and assume that* $1 \leqslant p \leqslant \infty$. *Then*

$$\| y^r V_\beta(y)\, q_n^{(s)}(y)\|_p \leqslant c(p, r, s)\, n^{\max(2r, s) - r} \, \| V_\beta(y)\, q_n(y)\|_p.$$

Proof. Let $\|f(x)\|_p^*$ denote the L_p norm of $f(x)$ with respect to the measure $|x|\, dx$ on $(-\infty, \infty)$. It is easy to see that for any polynomial q_n

$$\| W_\beta(x)\, q_n'(x)\|_p^* \leqslant c(p)\, n^{1/2} \, \| W_\beta(x)\, q_n(x)\|_p^*. \tag{9}$$

For $p = \infty$ this assertion trivially follows from [2, (29)], whereas for $1 \leqslant p < \infty$ it is a consequence of (1) and [3, (3.2.1)].

We now proceed with the proof of the theorem, starting with the case $2r \geqslant s$ (whence $\max(2r, s) - r = r$). The proof is by induction on s. Let $y = x^2$ and $p_n(x) = q_n(x^2)$; then if $q_n'(y) = (d/dy)\, q_n(y)$ and $p_n'(x) = (d/dx)\, p_n(x)$, it is clear that $y^{1/2} q_n'(y) = (1/2)\, p_n'(x)$. Since $W_\beta(x)\, p_n(x)$ is an even function, proceeding as in the proof of Theorem 6 we have

$$\| y^{1/2} V_\beta(y)\, q_n'(y)\|_p = \| W_\beta(x)\; p_n'(x)\|_p^*,$$

whence from (9) we infer that if $p = 1$ or $p = \infty$,

$$\| y^{1/2} V_\beta(y)\, q_n'(y)\|_p \leqslant c(p)\, n^{1/2}\, \| W_\beta(x)\, p_n(x)\|_p^* = c(p)\, n^{1/2}\, \| V_\beta(y)\, q_n(y)\|_p. \tag{10}$$

If $2r \geqslant 1$, applying Theorem 7 and then (10) we see that

$$\begin{aligned} \| y^r V_\beta(y)\, q_n'(y)\|_p &= \| y^{(r-1/2)+1/2} V_\beta(y)\, q_n'(y)\|_p \\ &\leqslant c(p, r)\, n^{r-1/2}\, \| y^{1/2} V_\beta(y)\, q_n'(y)\|_p \\ &\leqslant c(r, p)\, n^{r-1/2} n^{1/2}\, \| V_\beta(y)\, q_n(y)\|_p. \end{aligned}$$

We have therefore proved the assertion for $s = 1$.

Before proving the inductive step note that if s is a natural number, applying (9) we have

$$\begin{aligned} \| y^{1/2} V_\beta(y) [\, y^{s/2} q_n(y)]' \| &= \| W_\beta(x) [x^s p_n(x)]' \|_p^* \\ &< c(p, s)\, n^{1/2}\, \| W_\beta(x) x^s p_n(x)\|_p^* \\ &= c(p, s)\, n^{1/2}\, \| y^{s/2} V_\beta(y)\, q_n(y)\|_p. \end{aligned} \tag{11}$$

To prove the inductive step we proceed as follows: Since

$$y^{s/2} q_n^{(s+1)}(y) = [\, y^{s/2} q_n^{(s)}(y)]' - (s/2)\, y^{s/2-1} q_n^{(s)}(y), \tag{12}$$

it is clear that

$$\begin{aligned} &\| y^{(s+1)/2} V_\beta(y)\, q_n^{(s+1)}(y)\|_p \\ &\quad \leqslant \| y^{1/2} V_\beta(y) [\, y^{s/2} q_n^{(s)}(y)]' \|_p + (s/2)\, \| y^{(s-1)/2} V_\beta(y)\, q_n^{(s)}(y)\|_p, \end{aligned}$$

whence from (11), Theorem 9, and the inductive hypothesis,

$$\begin{aligned} &\| y^{(s+1)/2} V_\beta(y)\, q_n^{(s+1)}(y)\|_p \\ &\quad \leqslant c(p, s)\, n^{1/2}\, \| y^{s/2} V_\beta(y)\, q_n^{(s)}(y)\|_p + c(s)\, n^{1/2}\, \| y^{s/2} V_\beta(y)\, q_n^{(s)}(y)\|_p \\ &\quad \leqslant c(p, s)\, n^{1/2} n^{s/2}\, \| V_\beta(y)\, q_n(y)\|_p + c(p, s)\, n^{(s+1)/2}\, \| V_\beta(y)\, q_n(y)\|_p \\ &\quad = c(p, s)\, n^{(s+1)/2}\, \| V_\beta(y)\, q_n(y)\|_p. \end{aligned}$$

If $2r \geqslant s+1$, applying Theorem 7 we therefore have

$$\begin{aligned}\| y^r V_\beta(y)\, q_n^{(s+1)}(y)\|_p &= \| y^{r-(s+1)/2}\, y^{(s+1)/2}\, V_\beta(y)\, q_n^{(s+1)}(y)\|_p \\ &\leqslant c(p,r,s)\, n^{r-(s+1)/2}\, \| y^{(s+1)/2} V_\beta(y)\, q_n^{(s+1)}(y)\|_p \\ &\leqslant c(p,r,s)\, n^r\, \| V_\beta(y)\, q_n(y)\|_p, \qquad (13)\end{aligned}$$

and the conclusion follows.

Assume now that $2r < s$ (whence $\max(2r, s) - r = s - r$). We proceed by induction. Assume first that $s = 1$ and $0 \leqslant 2r < 1$. Since $x^{-1} p_n'(x)$ is a polynomial of degree $2n-2$, applying Theorem 5 (with $\delta = 1 - 2r$) and (9), we see that for any integer $k \geqslant 0$, if $p_n(x) = x^k q_n(x^2)$,

$$\begin{aligned}\| y^r V_\beta(y) [y^{k/2} q_n(y)]' \|_p &= \| \, |x|^{2r} W_\beta(x)\, |x|^{-1}\, p_n'(x)\|_p^* \\ &\leqslant c(r)\, n^{(1-2r)/2}\, \| W_\beta(x)\; p_n'(x)\|_p^* \\ &\leqslant c(p,r,k)\, n^{(1-2r)/2} n^{1/2} \| W_\beta(x)\; p_n(x)\|_p^* \\ &= c(p,r,k)\, n^{1-r}\, \| V_\beta(y)\, y^{k/2} q_n(y)\|_p. \qquad (14)\end{aligned}$$

We now prove the inductive step. Assume $0 \leqslant r \leqslant (s+1)/2$. We consider two cases. If $r < s/2$, from the inductive hypothesis and (14)

$$\begin{aligned}\| y^r V_\beta(y)\, q_n^{(s+1)}(y)\|_p &= \| y^r V_\beta(y) [q_n'(y)]^{(s)} \|_p \\ &\leqslant c(p,r,s)\, n^{s-r}\, \| V_\beta(y)\, q_n'(y)\|_p \\ &\leqslant c(p,r,s)\, n^{s+1-r}\, \| V_\beta(y)\, q_n(y)\|_p.\end{aligned}$$

On the other hand, if $r \geqslant (s/2)$, it is clear that $r = (s/2) + \delta$ with $0 < \delta < \frac{1}{2}$; thus from (12) we infer that

$$\begin{aligned}&\| y^r V_\beta(y)\, q_n^{(s+1)}(y)\|_p \\ &\quad \leqslant \| y^\delta V_\beta(y) [y^{s/2} q_n^{(s)}(y)]' \|_p + (s/2)\, \| y^{r-1} V_\beta(y)\, q_n^{(s)}(y)\|_p.\end{aligned}$$

Applying (14) (with r replaced by δ) Theorem 9 and (13) (with $s+1$ replaced by s), we therefore have

$$\begin{aligned}\| y^r V_\beta(y)\, q_n^{(s+1)}(y)\|_p &\leqslant c(p,r,s)\, n^{1-\delta}\, \| y^{s/2} V_\beta(y)\, q_n^{(s)}(y)\|_p \\ &\quad + c(r,s)\, n^{1-\delta}\, \| y^{s/2} V_\beta(y)\, q_n^{(s)}(y)\|_p \\ &\leqslant c(p,r,s)\, n^{1-\delta} n^{s/2}\, \| V_\beta(y)\, q_n(y)\|_p \\ &\quad + c(p,r,s)\, n^{1-\delta} n^{s/2}\, \| V_\beta(y)\, q_n(y)\|_p.\end{aligned}$$

Since $1 - \delta + s/2 = s + 1 - r$ the conclusion follows. Q.E.D.

REFERENCES

1. R. A. ZALIK, Inequalities for weighted polynomials, *J. Approx. Theory* **37** (1983), 137–146.
2. G. FREUD, On direct and converse theorems in the theory of weighted polynomial approximation, *Math. Z.* **126** (1972), 123–134.
3. G. FREUD, On polynomial approximation with respect to general weights, *in* "Functional Analysis and Applications" (A. Dold and B. Eckman, Eds.), Lecture Notes in Math. No. 399, pp. 149–179, Springer-Verlag, New York, 1974.
4. H. N. MHASKAR, "Weighted Polynomial Approximation on the Whole Real Line and Related Topics", Ph. D. dissertation, Ohio State University, 1980.
5. H. N. MHASKAR, Weighted analogues of Nikolskii-type inequalities and their applications, *in* "Conference on harmonic analysis in honor of Antoni Zygmund" (Beckner, Calderón, Fefferman, and Jones, Eds.), Vol. II, pp. 783–801, Wadsworth International, Belmont (1983).
6. S. S. BONAN, "Weighted Mean Convergence of Lagrange Interpolation," Ph. D. dissertation, Ohio State University, 1982.
7. I. S. GRADSHTEYN AND M. RYZHIK, Tables of Integrals, Series, and Products, Academic Press, New York, 1980.
8. G. FREUD, A contribution to the problem of weighted polynomial approximation, *in* "Linear operators and approximation theory" (P. L. Butzer, J. P. Kahane, and B. Sz.-Nagy, Eds.), ISNM Vol. 20, Birkhäuser-Verlag, Basel, 1972.
9. A. ZYGMUND, "Trigonometric Series," 2nd Ed., Vols. I and II, Cambridge Univ. Press, Cambridge, 1968.
10. H. N. MHASKAR AND E. B. SAFF, Extremal problems for polynomials with exponential weights, *Trans. Amer. Math. Soc.* (in press).
11. G. FREUD, On two polynomial inequalities II, *Acta Math. Acad. Sci. Hungar.* **23** (1972), 137–145.
12. H. N. MHASKAR AND E. B. SAFF, Extremal problems for polynomials with Laguerre weights, *in* "Approximation Theory IV" (Chui, Schumaker and Ward, Eds.), pp. 619–624, Academic Press, New York, 1983.